M10/5/MATHL/HP1/ENG/TZ2/XX

22107205

International Baccalaureate®
Baccalauréat International
Bachillerato Internacional

MATHEMATICS
HIGHER LEVEL
PAPER 1

Wednesday 5 May 2010 (afternoon)

2 hours

Candidate session number

0	0							

INSTRUCTIONS TO CANDIDATES

- Write your session number in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Section A: answer all of Section A in the spaces provided.
- Section B: answer all of Section B on the answer sheets provided. Write your session number on each answer sheet, and attach them to this examination paper and your cover sheet using the tag provided.
- At the end of the examination, indicate the number of sheets used in the appropriate box on your cover sheet.
- Unless otherwise stated in the question, all numerical answers must be given exactly or correct to three significant figures.

2210-7205

0114

14 pages

Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

SECTION A

*Answer **all** the questions in the spaces provided. Working may be continued below the lines, if necessary.*

1. *[Maximum mark: 5]*

A continuous random variable X has the probability density function f given by

$$f(x) = \begin{cases} c(x - x^2), & 0 \le x \le 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine c. *[3 marks]*

(b) Find $\mathrm{E}(X)$. *[2 marks]*

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

2. *[Maximum mark: 6]*

(a) Express the quadratic $3x^2 - 6x + 5$ in the form $a(x+b)^2 + c$, where $a, b, c \in \mathbb{Z}$. *[3 marks]*

(b) Describe a sequence of transformations that transforms the graph of $y = x^2$ to the graph of $y = 3x^2 - 6x + 5$. *[3 marks]*

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

3. *[Maximum mark: 5]*

The three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are given by

$$\boldsymbol{a} = \begin{pmatrix} 2y \\ -3x \\ 2x \end{pmatrix},\ \boldsymbol{b} = \begin{pmatrix} 4x \\ y \\ 3-x \end{pmatrix},\ \boldsymbol{c} = \begin{pmatrix} 4 \\ -7 \\ 6 \end{pmatrix} \text{ where } x,\ y \in \mathbb{R}.$$

(a) If $\boldsymbol{a} + 2\boldsymbol{b} - \boldsymbol{c} = 0$, find the value of x and of y. *[3 marks]*

(b) Find the exact value of $|\boldsymbol{a} + 2\boldsymbol{b}|$. *[2 marks]*

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

4. *[Maximum mark: 4]*

A biased coin is weighted such that the probability of obtaining a head is $\frac{4}{7}$. The coin is tossed 6 times and X denotes the number of heads observed. Find the value of the ratio $\frac{\mathrm{P}(X=3)}{\mathrm{P}(X=2)}$.

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

5. *[Maximum mark: 7]*

Consider the matrices

$$\boldsymbol{A}=\begin{pmatrix}3 & -2\\ 5 & -4\end{pmatrix},\ \boldsymbol{B}=\begin{pmatrix}1 & 3\\ 2 & -2\end{pmatrix}.$$

(a) Find $\boldsymbol{BA}$. *[2 marks]*

(b) Calculate $\det(\boldsymbol{BA})$. *[2 marks]*

(c) Find $\boldsymbol{A}(\boldsymbol{A}^{-1}\boldsymbol{B}+2\boldsymbol{A}^{-1})\boldsymbol{A}$. *[3 marks]*

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

6. *[Maximum mark: 6]*

If x satisfies the equation $\sin\left(x+\frac{\pi}{3}\right)=2\sin x\sin\left(\frac{\pi}{3}\right)$, show that $11\tan x = a + b\sqrt{3}$, where a, $b \in \mathbb{Z}^{+}$.

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

7. *[Maximum mark: 8]*

The function f is defined by $f(x) = e^{x^2 - 2x - 1.5}$.

(a) Find $f'(x)$. *[2 marks]*

(b) You are given that $y = \frac{f(x)}{x-1}$ has a local minimum at $x = a$, $a > 1$. Find the value of a. *[6 marks]*

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

8. *[Maximum mark: 7]*

The normal to the curve $xe^{-y} + e^{y} = 1 + x$, at the point $(c\,,\ \ln c)$, has a y-intercept $c^2 + 1$.

Determine the value of c.

. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .
. . . .

9. *[Maximum mark: 6]*

Find the value of $\int_0^1 t \ln(t+1)\, \mathrm{d}t$.

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

10. *[Maximum mark: 6]*

A function f is defined by $f(x)=\dfrac{2x-3}{x-1}$, $x\neq 1$.

(a) Find an expression for $f^{-1}(x)$. *[3 marks]*

(b) Solve the equation $\left|f^{-1}(x)\right|=1+f^{-1}(x)$. *[3 marks]*

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

. . . .

SECTION B

*Answer **all** the questions on the answer sheets provided. Please start each question on a new page.*

11. *[Maximum mark: 10]*

(a) Consider the following sequence of equations.

$$1\times 2=\frac{1}{3}(1\times 2\times 3),$$
$$1\times 2+2\times 3=\frac{1}{3}(2\times 3\times 4),$$
$$1\times 2+2\times 3+3\times 4=\frac{1}{3}(3\times 4\times 5),$$
$$\ldots .$$

(i) Formulate a conjecture for the n^{th} equation in the sequence.

(ii) Verify your conjecture for $n=4$. *[2 marks]*

(b) A sequence of numbers has the n^{th} term given by $u_n=2^n+3,\ n\in\mathbb{Z}^+$. Bill conjectures that all members of the sequence are prime numbers. Show that Bill's conjecture is false. *[2 marks]*

(c) Use mathematical induction to prove that $5\times 7^n+1$ is divisible by 6 for all $n\in\mathbb{Z}^+$. *[6 marks]*

12. *[Maximum mark: 19]*

(a) Consider the vectors $\boldsymbol{a} = 6\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$, $\boldsymbol{b} = -3\boldsymbol{j} + 4\boldsymbol{k}$.

(i) Find the cosine of the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

(ii) Find $\boldsymbol{a} \times \boldsymbol{b}$.

(iii) **Hence** find the Cartesian equation of the plane Π containing the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and passing through the point $(1, 1, -1)$.

(iv) The plane Π intersects the x-y plane in the line l. Find the area of the finite triangular region enclosed by l, the x-axis and the y-axis. *[11 marks]*

(b) Given two vectors $\boldsymbol{p}$ and $\boldsymbol{q}$,

(i) show that $\boldsymbol{p} \cdot \boldsymbol{p} = |\boldsymbol{p}|^2$;

(ii) hence, or otherwise, show that $|\boldsymbol{p} + \boldsymbol{q}|^2 = |\boldsymbol{p}|^2 + 2\boldsymbol{p} \cdot \boldsymbol{q} + |\boldsymbol{q}|^2$;

(iii) deduce that $|\boldsymbol{p} + \boldsymbol{q}| \leq |\boldsymbol{p}| + |\boldsymbol{q}|$. *[8 marks]*

13. *[Maximum mark: 16]*

Consider $\omega = \cos\left(\frac{2\pi}{3}\right) + \mathrm{i}\sin\left(\frac{2\pi}{3}\right)$.

(a) Show that

(i) $\omega^3 = 1$;

(ii) $1 + \omega + \omega^2 = 0$. *[5 marks]*

(b) (i) Deduce that $\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{2\pi}{3}\right)} + \mathrm{e}^{\mathrm{i}\left(\theta + \frac{4\pi}{3}\right)} = 0$.

(ii) Illustrate this result for $\theta = \frac{\pi}{2}$ on an Argand diagram. *[4 marks]*

(c) (i) Expand and simplify $F(z) = (z-1)(z-\omega)(z-\omega^2)$ where z is a complex number.

(ii) Solve $F(z) = 7$, giving your answers in terms of ω. *[7 marks]*

14. *[Maximum mark: 15]*

Throughout this question x satisfies $0 \le x < \frac{\pi}{2}$.

(a) Solve the differential equation $\sec^2 x \frac{\mathrm{d}y}{\mathrm{d}x} = -y^2$, where $y = 1$ when $x = 0$.

Give your answer in the form $y = f(x)$. *[7 marks]*

(b) (i) Prove that $1 \le \sec x \le 1 + \tan x$.

(ii) Deduce that $\frac{\pi}{4} \le \int_0^{\frac{\pi}{4}} \sec x \,\mathrm{d}x \le \frac{\pi}{4} + \frac{1}{2}\ln 2$. *[8 marks]*